

S_3 -COVERS OF SCHEMES

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ABSTRACT. We analyze flat S_3 -covers, attempting to create structures parallel to those found in the abelian theory. We use an initial local analysis as a guide in finding a global description.

1. INTRODUCTION

Given a finite group G , a G -cover of a scheme X is a scheme Y together with a faithful G -action on Y and a finite G -equivariant morphism $\pi : Y \rightarrow X$ which identifies X with the geometric quotient Y/G . If one considers only schemes over a fixed algebraically closed field k of characteristic prime to the order of G , then to each G -cover $\pi : Y \rightarrow X$ there is a decomposition $\pi_*\mathcal{O}_Y = \bigoplus_{\rho \in G^\vee} \mathcal{F}_\rho$, where \mathcal{F}_ρ is an $\mathcal{O}_X[G]$ -module with G -action closely related to the irreducible representation ρ . Under suitable additional hypotheses (e.g., X, Y integral and Noetherian, π flat), the sheaf \mathcal{F}_ρ is locally free of rank equal to $(\dim \rho)^2$. Conversely, to construct a cover given an appropriate collection of locally free $\mathcal{O}_X[G]$ -modules $\{\mathcal{F}_\rho\}_\rho$, one must define a commutative, associative $\mathcal{O}_X[G]$ -algebra structure on $\mathcal{A} = \bigoplus_\rho \mathcal{F}_\rho$. One then obtains the G -cover $\pi : \operatorname{Spec}_X \mathcal{A} \rightarrow X$.

The theory for abelian groups was analyzed by Pardini in [9]. In this case, the decomposition runs over the irreducible characters of G , and the $\mathcal{O}_X[G]$ -submodule \mathcal{F}_χ is the invertible χ -eigensheaf of $\pi_*\mathcal{O}_Y$, defined by the collections of sections on which the group acts as multiplication by the character χ . The algebra structure on $\pi_*\mathcal{O}_Y$ is determined by a compatible collection of morphisms $\{\mathcal{F}_\chi \otimes \mathcal{F}_{\chi'} \rightarrow \mathcal{F}_{\chi\chi'}\}_{\chi, \chi'}$, or equivalently, by a collection of global sections of $\{\mathcal{F}_\chi^{-1} \otimes \mathcal{F}_{\chi'}^{-1} \otimes \mathcal{F}_{\chi\chi'}\}_{\chi, \chi'}$. These sections are closely related to the branch divisor of the cover: given invertible sheaves $\{\mathcal{F}_\chi\}_\chi$, to construct a G -cover one may replace the explicit definition of the algebra structure with a specification of the branching data. As long as a “covering condition” is satisfied, one so builds a G -cover.

A key aspect of the abelian theory is that it allows one to understand the geometry of a covering scheme in terms of the geometry of the (usually simpler) base scheme. For example, if X is a surface, one can use geometrically interesting configurations of curves in X to construct new surfaces whose intrinsic geometry reflects the geometry of the configuration. This strategy has proven remarkably fruitful. For example, a standard result in the theory of complex surfaces is the Bogomolov-Miyaoka-Yau inequality, which states that $\frac{K_X^2}{\chi(\mathcal{O}_X)} \leq 9$ for any smooth, complex surface X of general type ([1],[8],[10]). This inequality is sharp, and Hirzebruch produced examples of equality by constructing abelian covers of \mathbb{P}^2 branched over “extreme” configurations of lines [5]. The inequality is known to fail

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in positive characteristic [6], and infinite families of counterexamples can be produced using abelian covers branched over configurations of lines occurring only in positive characteristic [3]. A second example of this strategy can be found in [4], in which a similar construction with abelian covers was employed to prove a higher-dimensional analogue of Belyi's theorem.

The situation in the nonabelian case is more complicated, perhaps even intractable in general, but the permutation group S_3 is within reach. The aim here is to create structures parallel to those found in the abelian case, to the extent possible, with the hope of eventually extending the applications of the abelian theory to this nonabelian setting. To realize this goal, we first proceed with a local analysis of S_3 -covers, which we subsequently use as a guide in studying the global situation. The precise statement of the main result (Theorem 4.14) requires some preliminary definitions, but can be roughly stated as follows:

1.1. Main Theorem. — *Let X be an integral, Noetherian scheme over an integral domain R in which 6 is invertible. Then the collection of all flat S_3 -covers of X is parameterized by the following data:*

- (i) *an invertible sheaf \mathcal{L} on X , on which S_3 acts via the sign character;*
- (ii) *a locally free $\mathcal{O}_X[S_3]$ -module \mathcal{E} , on which S_3 acts through its two-dimensional representation;*
- (iii) *a module, $\text{Build}_X(\mathcal{L}, \mathcal{E})$, parameterizing the “building data” which define commutative, associative algebra structures on $\mathcal{A} := \mathcal{O}_X \oplus \mathcal{L} \oplus \mathcal{E}$ compatible with the given S_3 -actions; i.e., the data required to construct a cover of the form $\pi : \text{Spec}_X \mathcal{A} \rightarrow X$.*

Acknowledgments. The current work grew out of [2], simplifying the local analysis contained in the latter and extending the analysis to the global situation. I am indebted to R. Vakil for his constant guidance and advice, both on the former project and in the present one.

2. PRELIMINARY ANALYSIS

Fix a presentation

$$S_3 = \langle \sigma, \tau \mid \sigma^3 = \tau^2 = e, \quad \tau\sigma = \sigma^2\tau \rangle.$$

Let R be an integral domain in which 6 is invertible. The group ring $R[S_3]$ can be decomposed (as a free R -module) as $R[S_3] = C_1 \oplus C_2 \oplus C_3$, where

$$\begin{aligned} C_1 &= \text{span}_R\{e + \sigma + \sigma^2 + \tau + \sigma\tau + \sigma^2\tau\} = \{v \in R[S_3] \mid g \cdot v = v, \forall g \in S_3\}, \\ C_2 &= \text{span}_R\{e + \sigma + \sigma^2 - \tau - \sigma\tau - \sigma^2\tau\} = \{v \in R[S_3] \mid g \cdot v = \text{sgn}(g)v, \forall g \in S_3\}, \end{aligned}$$

and where a basis for C_3 is given by the four vectors

$$\begin{aligned} u_{11} &= -e + \sigma + \tau - \sigma^2\tau \\ u_{12} &= -\sigma + \sigma^2 - \tau + \sigma\tau \\ u_{21} &= -e + \sigma^2 + \tau - \sigma\tau \\ u_{22} &= e - \sigma + \sigma\tau - \sigma^2\tau. \end{aligned}$$

Under this decomposition of $R[S_3]$, we have $e = e_1 + e_2 + e_3$, where

$$\begin{aligned} e_1 &= \frac{1}{6}(e + \sigma + \sigma^2 + \tau + \sigma\tau + \sigma^2\tau), \\ e_2 &= \frac{1}{6}(e + \sigma + \sigma^2 - \tau - \sigma\tau - \sigma^2\tau), \\ e_3 &= \frac{1}{3}(2e - \sigma - \sigma^2). \end{aligned}$$

Note that each e_i is in the center of $R[S_3]$ and satisfies $e_i e_j = \delta_{ij} e_i$. We also have a (non-equivariant) decomposition $C_3 = C_{31} \oplus C_{32}$, where $C_{3i} = \text{span}_R\{u_{i1}, u_{i2}\}$. Under this decomposition, e_3 decomposes as $e_3 = e_{31} + e_{32}$, where

$$\begin{aligned} e_{31} &= \frac{1}{3}(e - \sigma + \sigma\tau - \sigma^2\tau), \\ e_{32} &= \frac{1}{3}(e - \sigma^2 - \sigma\tau + \sigma^2\tau), \end{aligned}$$

which also satisfy $e_{3i} e_{3j} = \delta_{ij} e_{3i}$ (but are not central).

Suppose X is a scheme over R , and \mathcal{F} is an $\mathcal{O}_X[S_3]$ -module, with action explicitly given by a group homomorphism

$$\mu : S_3 \rightarrow \text{Aut}_{\mathcal{O}_X}(\mathcal{F}).$$

Extend this morphism R -linearly to a ring homomorphism

$$\mu : R[S_3] \rightarrow \text{End}_{\mathcal{O}_X}(\mathcal{F}).$$

It is immediate that, given any $f \in \text{End}_{\mathcal{O}_X}(\mathcal{F})$ satisfying $f \circ f = f$, the image presheaf of f is automatically a sheaf. Let $\mathcal{F}_i \subset \mathcal{F}$ denote the image (pre)sheaf of $\mu(e_i)$. Note that \mathcal{F}_i is characterized by the property that $\mu(e_j)$ acts as $\delta_{ij} \text{Id}_{\mathcal{F}_i}$. (In particular, \mathcal{F}_1 is the subsheaf of invariant sections of \mathcal{F} , and \mathcal{F}_2 is the subsheaf of sections on which S_3 acts as multiplication by the sign character.) Since $\mu(e) = \text{Id}_{\mathcal{F}}$, and since e decomposes as $e = e_1 + e_2 + e_3$ with elements e_i satisfying the aforementioned properties, the morphisms $\mu(e_i)$ induce an S_3 -equivariant direct sum decomposition

$$\mathcal{F} = \mathcal{F}_1 \oplus \mathcal{F}_2 \oplus \mathcal{F}_3.$$

We can further decompose the sheaf \mathcal{F}_3 , using the decomposition $e_3 = e_{31} + e_{32}$. We obtain an \mathcal{O}_X -module direct sum decomposition

$$\mathcal{F}_3 = \mathcal{F}_{31} \oplus \mathcal{F}_{32},$$

where \mathcal{F}_{3i} is the image (pre)sheaf associated to $\mu(e_{3i})|_{\mathcal{F}_3}$. Note that these subsheaves are not invariant under the group action. Indeed, observe that

$$\tau e_{31} = \frac{1}{3}(\tau - \sigma^2\tau + \sigma^2 - \sigma) = e_{32}\tau,$$

so $\mu(\tau)|_{\mathcal{F}_3}$ is an automorphism interchanging the summands. We can therefore write $\mathcal{F}_3 = \mathcal{F}_{31} \oplus \tau\mathcal{F}_{31}$.

Suppose $\pi : Y \rightarrow X$ is a flat S_3 -cover of R -schemes. Then $\pi_*\mathcal{O}_Y$ is a locally free $\mathcal{O}_X[S_3]$ -module of rank 6, with $(\pi_*\mathcal{O}_Y)^{S_3} = \mathcal{O}_X$. By the above, we have an induced $\mathcal{O}_X[S_3]$ -module direct sum decomposition $\pi_*\mathcal{O}_Y = \mathcal{O}_X \oplus \mathcal{F}_2 \oplus \mathcal{F}_3$ (as well as an \mathcal{O}_X -module decomposition $\mathcal{F}_3 = \mathcal{F}_{31} \oplus \tau\mathcal{F}_{31}$). If we assume X, Y are integral, Noetherian R -schemes, then it follows that \mathcal{F}_2 and \mathcal{F}_{31} are also locally free, of ranks 1 and 2, respectively.

Since a finite morphism is affine, it follows that $Y = \operatorname{Spec}_X(\mathcal{O}_X \oplus \mathcal{F}_2 \oplus \mathcal{F}_3)$. Thus, to construct a flat S_3 -cover of a given scheme X , one needs the following data:

- (i) An invertible sheaf \mathcal{L} , on which S_3 acts via the sign character;
- (ii) A locally free \mathcal{O}_X -module \mathcal{E} of rank 4, together with an S_3 -action such that (under the induced $R[S_3]$ -action), e_i acts as $\delta_{3i}\operatorname{Id}_{\mathcal{E}}$; and
- (iii) A commutative, associative \mathcal{O}_X -algebra structure on $\mathcal{A} := \mathcal{O}_X \oplus \mathcal{L} \oplus \mathcal{E}$ compatible with the given S_3 -action.

We aim to precisely describe data (iii), given data (i) and (ii). To that end, suppose we are given such \mathcal{O}_X -modules \mathcal{L} and \mathcal{E} .

2.1. Lemma. — *A commutative \mathcal{O}_X -algebra structure on \mathcal{A} compatible with the given S_3 -actions is defined by a triple of \mathcal{O}_X -module morphisms*

$$\alpha : S^2\mathcal{L} \rightarrow \mathcal{O}_X, \quad \beta : \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{E} \rightarrow \mathcal{E}, \quad \gamma : S^2\mathcal{E} \rightarrow \mathcal{A}.$$

Proof. A priori, an algebra structure is given by an \mathcal{O}_X -module morphism $\mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{A} \rightarrow \mathcal{A}$, which is equivalent to an \mathcal{O}_X -module morphism

$$\begin{aligned} &(\mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{O}_X) \oplus (\mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{L}) \oplus (\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{O}_X) \oplus (\mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{E}) \oplus (\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{O}_X) \\ &\oplus (\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{L}) \oplus (\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{E}) \oplus (\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{L}) \oplus (\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{E}) \rightarrow \mathcal{A}. \end{aligned}$$

The first coordinate of this morphism must be given by the algebra structure on \mathcal{O}_X , and the following four coordinates must be given by the left and right \mathcal{O}_X -module structures on \mathcal{L} and \mathcal{E} , respectively. Commutativity requires the sixth and ninth coordinates to factor through the canonical morphisms to the corresponding symmetric products:

$$\begin{array}{ccc} \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{L} & \longrightarrow & \mathcal{A}, \\ \text{can} \downarrow & \nearrow & \\ S^2\mathcal{L} & & \end{array} \quad \begin{array}{ccc} \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{E} & \longrightarrow & \mathcal{A}. \\ \text{can} \downarrow & \nearrow \gamma & \\ S^2\mathcal{E} & & \end{array}$$

Similarly, the seventh and eighth coordinates must agree after the canonical braid isomorphism:

$$\begin{array}{ccc} \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{E} & \longrightarrow & \mathcal{A}. \\ \cong \downarrow \text{can} & \nearrow & \\ \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{L} & & \end{array}$$

Compatibility with the S_3 -action implies that S_3 acts as $(\text{sgn})^2 = \text{Id}$ on the image of $S^2\mathcal{L} \rightarrow \mathcal{A}$, and so the morphism must factor through the subsheaf $\mathcal{O}_X \subset \mathcal{A}$ of invariant sections:

$$\begin{array}{ccc} S^2\mathcal{L} & \longrightarrow & \mathcal{A} \\ & \searrow \alpha & \uparrow \\ & & \mathcal{O}_X \end{array}$$

Similarly, $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{E} \rightarrow \mathcal{A}$ must factor through $\mathcal{E} \subset \mathcal{A}$:

$$\begin{array}{ccc} \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{E} & \longrightarrow & \mathcal{A} \\ & \searrow \beta & \uparrow \\ & & \mathcal{E} \end{array}$$

(Suppose $U \subset X$ is an affine open and $t \in \mathcal{L}(U), x \in \mathcal{E}(U)$. Then

$$\begin{aligned} \mu(e_3)(t \otimes x) &= \frac{1}{3}(2\mu(e)(t \otimes x) - \mu(\sigma)(t \otimes x) - \mu(\sigma^2)(t \otimes x)) \\ &= \frac{1}{3}(2(t \otimes x) - (\mu(\sigma)t) \otimes (\mu(\sigma)x) - (\mu(\sigma^2)t) \otimes (\mu(\sigma^2)x)) \\ &= \frac{1}{3}(t \otimes (2x) - t \otimes (\mu(\sigma)x) - t \otimes (\mu(\sigma^2)x)) \\ &= t \otimes (\mu(e_3)x) \\ &= t \otimes x. \end{aligned}$$

Similarly, one sees $\mu(e_1)(t \otimes x) = t \otimes (\mu(e_2)x) = 0$ and $\mu(e_2)(t \otimes x) = t \otimes (\mu(e_1)x) = 0$. \square

Of course, not every such triple of morphisms will satisfy all of the necessary compatibilities with the S_3 -action, nor will it necessarily define an associative algebra structure. To precisely identify which triples do satisfy these conditions, we first analyze the local situation.

3. LOCAL ANALYSIS

Continuing the notation above, let $\mathcal{E} = \mathcal{E}' \oplus \tau\mathcal{E}'$ be the \mathcal{O}_X -module decomposition induced by $e_3 = e_{31} + e_{32}$. Let $U \subset X$ be any affine open such that $\mathcal{L}(U), \mathcal{E}'(U)$ are free $\mathcal{O}_X(U)$ -modules of ranks 1, 2, respectively. For notational simplicity, write $L = \mathcal{L}(U), E = \mathcal{E}(U), E' = \mathcal{E}'(U), B = \mathcal{O}_X(U)$, and $A = \mathcal{A}(U)$. Let $\{v_1, v_2\}$ be any basis for E' (and so $\{v_1, v_2, \tau v_1, \tau v_2\}$ is a basis for E).

3.1. Lemma. — *The S_3 -action on E satisfies:*

- (i) $\sigma v_i = -\tau v_i$
- (ii) $\sigma^2 v_i = -v_i + \tau v_i$
- (iii) $\sigma \tau v_i = v_i - \tau v_i$
- (iv) $\sigma^2 \tau v_i = -v_i$.

Proof. By direct calculation, one sees

$$\begin{aligned}\sigma e_{31} &= \frac{1}{3}(\sigma - \sigma^2 + \sigma^2\tau - \tau) = -\tau e_{31}, \\ \sigma^2 e_{31} &= \frac{1}{3}(\sigma^2 - e + \tau - \sigma\tau) = (-e + \tau)e_{31}.\end{aligned}$$

It follows that $\mu(\sigma)|_{E'} = \mu(-\tau)|_{E'}$ and $\mu(\sigma^2)|_{E'} = \mu(-e + \tau)|_{E'}$, which prove (i) and (ii). Properties (iii) and (iv) then immediately follow from the relations $\sigma\tau = \tau\sigma^2$ and $\sigma^2\tau = \tau\sigma$. \square

Let $t \in L$ be any generator. To understand the algebra structure, we need to analyze (the images of) the following products (with tensor symbols suppressed):

- $t^2 \in B$
- $tv_i, t \cdot \tau v_i \in E$, for $i = 1, 2$
- $v_i v_j, v_i \cdot \tau v_j, \tau v_i \cdot \tau v_j \in A$, for $i, j = 1, 2$

3.2. Lemma. — *With respect to the basis $\{1, t, v_1, v_2, \tau v_1, \tau v_2\}$ for A , the commutative $B[S_3]$ -algebra structures on A are precisely those of the form*

$$\begin{aligned}t^2 &= a \\ tv_1 &= b_1 v_1 + b_2 v_2 - 2b_1 \tau v_1 - 2b_2 \tau v_2 \\ tv_2 &= c_1 v_1 + c_2 v_2 - 2c_1 \tau v_1 - 2c_2 \tau v_2 \\ t \cdot \tau v_1 &= 2b_1 v_1 + 2b_2 v_2 - b_1 \tau v_1 - b_2 \tau v_2 \\ t \cdot \tau v_2 &= 2c_1 v_1 + 2c_2 v_2 - c_1 \tau v_1 - c_2 \tau v_2 \\ v_1^2 &= d_1 + d_3 v_1 + d_4 v_2 - 2d_3 \tau v_1 - 2d_4 \tau v_2 \\ v_1 v_2 &= e_1 + e_3 v_1 + e_4 v_2 - 2e_3 \tau v_1 - 2e_4 \tau v_2 \\ v_2^2 &= f_1 + f_3 v_1 + f_4 v_2 - 2f_3 \tau v_1 - 2f_4 \tau v_2 \\ v_1 \cdot \tau v_1 &= \frac{1}{2}d_1 - d_3 v_1 - d_4 v_2 - d_3 \tau v_1 - d_4 \tau v_2 \\ v_1 \cdot \tau v_2 &= \frac{1}{2}e_1 + h_2 t + h_3 v_1 + h_4 v_2 - e_3 \tau v_1 - e_4 \tau v_2 \\ v_2 \cdot \tau v_1 &= \frac{1}{2}e_1 - h_2 t - e_3 v_1 - e_4 v_2 + h_3 \tau v_1 + h_4 \tau v_2 \\ v_2 \cdot \tau v_2 &= \frac{1}{2}f_1 - f_3 v_1 - f_4 v_2 - f_3 \tau v_1 - f_4 \tau v_2 \\ (\tau v_1)^2 &= d_1 - 2d_3 v_1 - 2d_4 v_2 + d_3 \tau v_1 + d_4 \tau v_2 \\ (\tau v_1)(\tau v_2) &= e_1 - 2e_3 v_1 - 2e_4 v_2 + e_3 \tau v_1 + e_4 \tau v_2 \\ (\tau v_2)^2 &= f_1 - 2f_3 v_1 - 2f_4 v_2 + f_3 \tau v_1 + f_4 \tau v_2,\end{aligned}$$

for some $a, b_i, c_i, d_i, e_i, f_i, h_i \in B$.

Proof. Note that compatibility with the action of τ requires

$$\tau(tv_i) = (\tau t)(\tau v_i) = -t \cdot \tau v_i,$$

and so $t \cdot \tau v_i$ is determined by tv_i . Similarly, $\tau v_i \cdot \tau v_j$ is determined by $v_i v_j$, and $v_2 \cdot \tau v_1$ is determined by $v_1 \cdot \tau v_2$. So, suppose

$$\begin{aligned}
t^2 &= a \\
tv_1 &= b_1 v_1 + b_2 v_2 + b_3 \tau v_1 + b_4 \tau v_2 \\
tv_2 &= c_1 v_1 + c_2 v_2 + c_3 \tau v_1 + c_4 \tau v_2 \\
v_1^2 &= d_1 + d_2 t + d_3 v_1 + d_4 v_2 + d_5 \tau v_1 + d_6 \tau v_2 \\
v_1 v_2 &= e_1 + e_2 t + e_3 v_1 + e_4 v_2 + e_5 \tau v_1 + e_6 \tau v_2 \\
v_2^2 &= f_1 + f_2 t + f_3 v_1 + f_4 v_2 + f_5 \tau v_1 + f_6 \tau v_2 \\
v_1 \cdot \tau v_1 &= g_1 + g_2 t + g_3 v_1 + g_4 v_2 + g_5 \tau v_1 + g_6 \tau v_2 \\
v_1 \cdot \tau v_2 &= h_1 + h_2 t + h_3 v_1 + h_4 v_2 + h_5 \tau v_1 + h_6 \tau v_2 \\
v_2 \cdot \tau v_2 &= i_1 + i_2 t + i_3 v_1 + i_4 v_2 + i_5 \tau v_1 + i_6 \tau v_2,
\end{aligned}$$

for some $a, b_j, c_j, d_j, e_j, f_j, g_j, h_j, i_j \in B$. By our previous remark, we then must have

$$\begin{aligned}
t \cdot \tau v_1 &= -b_3 v_1 - b_4 v_2 - b_1 \tau v_1 - b_2 \tau v_2 \\
t \cdot \tau v_2 &= -c_3 v_1 - c_4 v_2 - c_1 \tau v_1 - c_2 \tau v_2 \\
(\tau v_1)^2 &= d_1 - d_2 t + d_5 v_1 + d_6 v_2 + d_3 \tau v_1 + d_4 \tau v_2 \\
(\tau v_1)(\tau v_2) &= e_1 - e_2 t + e_5 v_1 + e_6 v_2 + e_3 \tau v_1 + e_4 \tau v_2 \\
(\tau v_2)^2 &= f_1 - f_2 t + f_5 v_1 + f_6 v_2 + f_3 \tau v_1 + f_4 \tau v_2 \\
v_2 \cdot \tau v_1 &= h_1 - h_2 t + h_5 v_1 + h_6 v_2 + h_3 \tau v_1 + h_4 \tau v_2.
\end{aligned}$$

Such an algebra structure is now ensured to be compatible with the action of τ , and so compatibility with the full S_3 -action will follow from compatibility with the action of σ . We now impose this compatibility on each product. Using the equations above, together with Lemma 3.1, we compute

$$\sigma(tv_1) = b_3 v_1 + b_4 v_2 + (-b_1 - b_3) \tau v_1 + (-b_2 - b_4) \tau v_2,$$

and

$$\begin{aligned}
(\sigma t)(\sigma v_1) &= t \cdot \sigma v_1 = t \cdot (-\tau v_1) = -(t \cdot \tau v_1) \\
&= b_3 v_1 + b_4 v_2 + b_1 \tau v_1 + b_2 \tau v_2.
\end{aligned}$$

So, compatibility with σ requires $b_3 = -2b_1$ and $b_4 = -2b_2$. The corresponding computation for the relation $\sigma(tv_2) = (\sigma t)(\sigma v_2)$ requires $c_3 = -2c_1$ and $c_4 = -2c_2$.

Similarly, we compute

$$\sigma(v_1^2) = d_1 + d_2 t + d_5 v_1 + d_6 v_2 + (-d_3 - d_5) \tau v_1 + (-d_4 - d_6) \tau v_2,$$

and

$$\begin{aligned}
(\sigma v_1)^2 &= (-\tau v_1)^2 = (\tau v_1)^2 \\
&= d_1 - d_2 t + d_5 v_1 + d_6 v_2 + d_3 \tau v_1 + d_4 \tau v_2.
\end{aligned}$$

So, we must have $d_2 = 0$, $d_5 = -2d_3$, and $d_6 = -2d_4$. Similarly, the relation $\sigma(v_1 v_2) = \sigma(v_1) \sigma(v_2)$ requires $e_2 = 0$, $e_5 = -2e_3$, and $e_6 = -2e_4$, while the relation $\sigma(v_2)^2 = (\sigma v_2)^2$ requires $f_2 = 0$, $f_5 = -2f_3$ and $f_6 = -2f_4$.

Lastly, we compute

$$\sigma(v_1 \cdot \tau v_1) = g_1 + g_2 t + g_5 v_1 + g_6 v_2 + (-g_3 - g_5) \tau v_1 + (-g_4 - g_6) \tau v_2,$$

and

$$\begin{aligned} (\sigma v_1)(\sigma \tau v_1) &= (-\tau v_1)(v_1 - \tau v_1) = -(v_1 \cdot \tau v_1) + (\tau v_1)^2 \\ &= (-g_1 + d_1) + (-g_2 - d_2)t + (-g_3 + d_5)v_1 + (-g_4 + d_6)v_2 \\ &\quad + (-g_5 + d_3)\tau v_1 + (-g_6 + d_4)\tau v_2. \end{aligned}$$

From this (and the previously obtained relations), we deduce $g_1 = \frac{1}{2}d_1$, $g_2 = 0$, $g_3 = g_5 = -d_3$, and $g_4 = g_6 = -d_4$. Similarly, the relation $\sigma(v_1 \cdot \tau v_2) = (\sigma v_1)(\sigma \tau v_2)$ requires $h_1 = \frac{1}{2}e_1$, $h_5 = -e_3$, and $h_6 = -e_4$, while the relation $\sigma(v_2 \cdot \tau v_2) = (\sigma v_2)(\sigma \tau v_2)$ requires $i_1 = \frac{1}{2}f_1$, $i_2 = 0$, $i_3 = i_5 = -f_3$ and $i_4 = i_6 = -f_4$. \square

Notice we specifically omitted any mention of associativity. Indeed, the associativity relations impose many additional conditions on the system.

3.3. Proposition. — *With respect to the basis $\{1, t, v_1, v_2, \tau v_1, \tau v_2\}$ for A , every commutative, associative, integral $B[S_3]$ -algebra structure on A is of the form*

$$\begin{aligned} t^2 &= -3b_1^2 - 3b_2c_1 \\ \tau v_1 &= b_1v_1 + b_2v_2 - 2b_1\tau v_1 - 2b_2\tau v_2 \\ \tau v_2 &= c_1v_1 - b_1v_2 - 2c_1\tau v_1 + 2b_1\tau v_2 \\ t \cdot \tau v_1 &= 2b_1v_1 + 2b_2v_2 - b_1\tau v_1 - b_2\tau v_2 \\ t \cdot \tau v_2 &= 2c_1v_1 - 2b_1v_2 - c_1\tau v_1 + b_1\tau v_2 \\ v_1^2 &= 6(d_3^2 - d_4f_4) + d_3v_1 + d_4v_2 - 2d_3\tau v_1 - 2d_4\tau v_2 \\ v_1v_2 &= 3(d_4f_3 - d_3f_4) - f_4v_1 - d_3v_2 + 2f_4\tau v_1 + 2d_3\tau v_2 \\ v_2^2 &= 6(f_4^2 - d_3f_3) + f_3v_1 + f_4v_2 - 2f_3\tau v_1 - 2f_4\tau v_2 \\ v_1 \cdot \tau v_1 &= 3(d_3^2 - d_4f_4) - d_3v_1 - d_4v_2 - d_3\tau v_1 - d_4\tau v_2 \\ v_1 \cdot \tau v_2 &= \frac{3}{2}(d_4f_3 - d_3f_4) + h_2t + f_4v_1 + d_3v_2 + f_4\tau v_1 + d_3\tau v_2 \\ v_2 \cdot \tau v_1 &= \frac{3}{2}(d_4f_3 - d_3f_4) - h_2t + f_4v_1 + d_3v_2 + f_4\tau v_1 + d_3\tau v_2 \\ v_2 \cdot \tau v_2 &= 3(f_4^2 - d_3f_3) - f_3v_1 - f_4v_2 - f_3\tau v_1 - f_4\tau v_2 \\ (\tau v_1)^2 &= 6(d_3^2 - d_4f_4) - 2d_3v_1 - 2d_4v_2 + d_3\tau v_1 + d_4\tau v_2 \\ (\tau v_1)(\tau v_2) &= 3(d_4f_3 - d_3f_4) + 2f_4v_1 + 2d_3v_2 - f_4\tau v_1 - d_3\tau v_2 \\ (\tau v_2)^2 &= 6(f_4^2 - d_3f_3) - 2f_3v_1 - 2f_4v_2 + f_3\tau v_1 + f_4\tau v_2, \end{aligned}$$

for some $b_1, b_2, c_1, d_3, d_4, f_3, f_4, h_2 \in B$ satisfying

- (i) $2b_1d_3 - b_2f_4 + c_1d_4 = 0$;
- (ii) $2b_1f_4 - b_2f_3 + c_1d_3 = 0$; and
- (iii) $(b_1^2 + b_2c_1)h_2 = \frac{3}{2}(b_1(d_4f_3 - d_3f_4) + b_2(f_4^2 - d_3f_3) + c_1(d_4f_4 - d_3^2))$.

Conversely, any multiplicative structure on A of the above form defines a commutative, associative (but possibly non-integral) $B[S_3]$ -algebra structure on A .

Proof. The proof consists of systematically imposing the third-order associativity conditions. Using Lemma 3.2, we compute

$$\begin{aligned}(t^2)v_1 &= av_1 \\ t(tv_1) &= (-3b_1^2 - 3b_2c_1)v_1 + (-3b_1b_2 - 3b_2c_2)v_2.\end{aligned}$$

Equating coefficients gives

$$\begin{aligned}(1) \quad & a = -3b_1^2 - 3b_2c_1 \\ (2) \quad & 0 = b_2(b_1 + c_2).\end{aligned}$$

Similarly, we compute

$$\begin{aligned}(t^2)v_2 &= av_2 \\ t(tv_2) &= (-3b_1c_1 - 3c_1c_2)v_1 + (-3b_2c_1 - 3c_2^2)v_2,\end{aligned}$$

and so must have

$$\begin{aligned}(3) \quad & 0 = c_1(b_1 + c_2) \\ (4) \quad & a = -3b_2c_1 - 3c_2^2.\end{aligned}$$

Note that the relations $(t^2)\tau v_1 = t(t\tau v_1)$ and $(t^2)\tau v_2 = t(t\tau v_2)$ immediately follow from the above relations and the compatibility with τ . Indeed, we have $(t^2)\tau v_1 = (-t)^2\tau v_1 = (\tau t)^2\tau v_1 = \tau((t^2)v_1) = \tau(t(tv_1)) = -t(-t \cdot \tau v_1) = t(t\tau v_1)$, and similarly for $(t^2)\tau v_2$.

We next compute

$$\begin{aligned}t(v_1^2) &= d_1t + (-3b_1d_3 - 3c_1d_4)v_1 + (-3b_2d_3 - 3c_2d_4)v_2, \\ (tv_1)v_1 &= (-2b_2h_2)t + (3b_1d_3 + b_2e_3 - 2b_2h_3)v_1 + (3b_1d_4 + b_2e_4 - 2b_2h_4)v_2,\end{aligned}$$

which implies

$$\begin{aligned}(5) \quad & d_1 = -2b_2h_2 \\ (6) \quad & 6b_1d_3 + b_2e_3 - 2b_2h_3 + 3c_1d_4 = 0 \\ (7) \quad & 3b_1d_4 + 3b_2d_3 + b_2e_4 - 2b_2h_4 + 3c_2d_4 = 0.\end{aligned}$$

We claim integrality of the algebra structure requires $c_2 = -b_1$. Indeed, suppose $c_2 \neq -b_1$. Then equations (2) and (3) imply $b_2 = c_1 = 0$. Equations (1) and (4) then become $-3c_2^2 = a = -3b_1^2$. By hypothesis, the algebra is integral, so $t^2 = a$ is nonzero, and hence it follows that b_1 and c_2 are both nonzero. Since their squares are equal and $c_2 \neq -b_1$, we must have $c_2 = b_1$. But then equations (5) – (7) imply $d_1 = d_3 = d_4 = 0$, and hence $v_1^2 = 0$, which violates integrality.

So, we must have $c_2 = -b_1$, and equations (1)-(7) now reduce to

$$\begin{aligned}
(8) \quad & a = -3b_1^2 - 3b_2c_1 \\
(9) \quad & d_1 = -2b_2h_2 \\
(10) \quad & 6b_1d_3 + b_2e_3 - 2b_2h_3 + 3c_1d_4 = 0 \\
(11) \quad & 3b_2d_3 + b_2e_4 - 2b_2h_4 = 0.
\end{aligned}$$

We now continue computing the associativity relations:

$$\begin{aligned}
t(v_1v_2) &= e_1t + (-3b_1e_3 - 3c_1e_4)v_1 + (-3b_2e_3 + 3b_1e_4)v_2 \\
(tv_1)v_2 &= (2b_1h_2)t + (3b_1e_3 + 3b_2f_3)v_1 + (3b_1e_4 + 3b_2f_4)v_2 \\
&\quad + (-2b_1e_3 - 2b_1h_3)\tau v_1 + (-2b_1e_4 - 2b_1h_4)\tau v_2.
\end{aligned}$$

It follows that

$$\begin{aligned}
(12) \quad & e_1 = 2b_1h_2 \\
(13) \quad & 2b_1e_3 + b_2f_3 + c_1e_4 = 0 \\
(14) \quad & b_2(e_3 + f_4) = 0 \\
(15) \quad & b_1(e_3 + h_3) = 0 \\
(16) \quad & b_1(e_4 + h_4) = 0.
\end{aligned}$$

We next compute:

$$\begin{aligned}
t(v_2^2) &= f_1t + (-3b_1f_3 - 3c_1f_4)v_1 + (-3b_2f_3 + 3b_1f_4)v_2 \\
(tv_2)v_2 &= (2c_1h_2)t + (3c_1e_3 - 3b_1f_3)v_1 + (3c_1e_4 - 3b_1f_4)v_2 \\
&\quad + (-2c_1e_3 - 2c_1h_3)\tau v_1 + (-2c_1e_4 - 2c_1h_4)\tau v_2,
\end{aligned}$$

which implies

$$\begin{aligned}
(17) \quad & f_1 = 2c_1h_2 \\
(18) \quad & c_1(e_3 + f_4) = 0 \\
(19) \quad & 2b_1f_4 - b_2f_3 - c_1e_4 = 0 \\
(20) \quad & c_1(e_3 + h_3) = 0 \\
(21) \quad & c_1(e_4 + h_4) = 0.
\end{aligned}$$

Now, again by integrality, we cannot have both b_1 and b_2 both zero (else $tv_1 = 0$), nor b_1 and c_1 both zero (else $tv_2 = 0$). In particular, equations (15) and (20) together imply $h_3 = -e_3$. Similarly, equations (16) and (21) imply $h_4 = -e_4$. Also observe that summing equations (13) and (19) yields the equation $b_1(e_3 + f_4)$, which together with equation (14) (or (18)) implies $e_3 = -f_4$.

We once again continue with the associativity relations. A calculation reveals the relations

$$t(v_1\tau v_1) = (tv_1)\tau v_1, \quad t(v_1\tau v_2) = (tv_1)\tau v_2, \quad t(v_2\tau v_2) = (tv_2)\tau v_2$$

all now hold (under the conditions summarized above). Compatibility with the action of τ then implies that the relations

$$t(\tau v_1)^2 = (t \cdot \tau v_1) \tau v_1, \quad t(\tau v_1 \cdot \tau v_2) = (t \cdot \tau v_1) \tau v_2, \quad t(\tau v_2)^2 = (t \cdot \tau v_2) \tau v_2, \quad t(v_2 \cdot \tau v_1) = (t v_2) \tau v_1$$

automatically follow from the previous associativity relations. So, we next compute

$$\begin{aligned} (v_1^2)v_2 &= (2d_3h_2)t + (-3d_3f_4 + 3d_4f_3)v_1 + (-2b_2h_2 + 3d_3e_4 + 3d_4f_4)v_2 \\ v_1(v_1v_2) &= (-2e_4h_2)t + (2b_1h_2 - 3d_3f_4 - 3e_4f_4)v_1 + (-3d_4f_4 + 3e_4^2)v_2, \end{aligned}$$

which implies

$$(22) \quad h_2(d_3 + e_4) = 0$$

$$(23) \quad 2b_1h_2 - 3d_4f_3 - 3e_4f_4 = 0$$

$$(24) \quad 2b_2h_2 - 3d_3e_4 - 6d_4f_4 + 3e_4^2 = 0.$$

Continuing, we compute

$$\begin{aligned} v_1(v_2^2) &= (-2f_4h_2)t + (2c_1h_2 + 3d_3f_3 - 3f_4^2)v_1 + (d_4f_2 + 3e_4f_4)v_1 \\ (v_1v_2)v_2 &= (-2f_4h_2)t + (3f_4^2 + 3e_4f_3)v_1 + (2b_1h_2)v_2, \end{aligned}$$

which implies

$$(25) \quad 2c_1h_2 + 3d_3f_3 - 3e_4f_3 - 6f_4^2 = 0.$$

Lastly, we compute

$$\begin{aligned} v_1^2(\tau v_1) &= (3b_2d_3h_2 - 3b_1d_4h_2) + (-d_4h_2)t + (3d_3^2 - 3d_4f_4)v_1 + (3d_3d_4 + 3d_4e_4)v_2 \\ &\quad + (-2b_2h_2 - 3d_3^2 + 3d_4f_4)\tau v_1 + (-3d_3d_4 - 3d_4e_4)\tau v_2 \\ v_1(v_1\tau v_1) &= (3b_2d_3h_2 - 3b_1d_4h_2) + (-d_4h_2)t + (-b_2h_2)v_1 \\ &\quad + (3d_3^2 - 3d_4f_4)\tau v_1 + (3d_3d_4 + 3d_4e_4)\tau v_2, \end{aligned}$$

which implies

$$(26) \quad b_2h_2 + 3d_3^2 - 3d_4f_4 = 0$$

$$(27) \quad d_4(d_3 + e_4) = 0.$$

Observe that if $e_4 \neq -d_3$, then equations (11), (22), and (27) imply $b_2 = h_2 = d_4 = 0$, which together with equation (26) imply $d_3 = 0$. But then $v_1^2 = 0$, which violates integrality. So, we must have $e_4 = -d_3$.

At this point, we've reached the statement of Proposition 3.3. Indeed, equations (23) – (25) (together with (5), (12), (17)) can all be immediately solved to give the constant coefficients of the terms $v_1^2, \dots, (\tau v_2)^2$, as well as combined to give relation (iii) of the proposition. A calculation verifies all remaining associativity relations are now satisfied. \square

3.4. Corollary. — Continuing the notation from Proposition 3.3, the ramification locus of $\pi : \text{Spec } A \rightarrow \text{Spec } B$ is the zero locus of the ideal generated by all 5×5 minors of the matrix

$$\begin{bmatrix} 2t & 0 & 0 & 0 & 0 \\ v_1 & t - b_1 & -b_2 & 2b_1 & 2b_2 \\ v_2 & -c_1 & t + b_1 & 2c_1 & -2b_1 \\ \tau v_1 & -2b_1 & -2b_2 & t + b_1 & b_2 \\ \tau v_2 & -2c_1 & 2b_1 & c_1 & t - b_1 \\ 0 & 2v_1 - d_3 & -d_4 & 2d_3 & 2d_4 \\ 0 & v_2 + f_4 & v_1 + d_3 & -2f_4 & -2d_3 \\ 0 & -f_3 & 2v_2 - f_4 & 2f_3 & 2f_4 \\ 0 & \tau v_1 + d_3 & d_4 & v_1 + d_3 & d_4 \\ -h_2 & \tau v_2 - f_4 & -d_3 & -f_4 & v_1 - d_3 \\ h_2 & -f_4 & \tau v_1 - d_3 & v_2 - f_4 & -d_3 \\ 0 & f_3 & \tau v_2 + f_4 & f_3 & v_2 + f_4 \\ 0 & 2d_3 & 2d_4 & 2\tau v_1 - d_3 & -d_4 \\ 0 & -2f_4 & -2d_3 & \tau v_2 + f_4 & \tau v_1 + d_3 \\ 0 & 2f_3 & 2f_4 & -f_3 & 2\tau v_2 - f_4 \end{bmatrix},$$

under the identification of A with the quotient of $B[1, t, v_1, v_2, \tau v_1, \tau v_2]$ by the ideal of relations generated by the equations of Proposition 3.3.

3.5. Corollary. — The multiplication in A is determined by a triple of morphisms

$$\phi : L \otimes E' \rightarrow E' \quad \psi : S^2 E' \rightarrow E' \quad \xi : E' \otimes \tau E' \rightarrow L.$$

If t is a generator for L and $\{v_1, v_2\}$ is a basis for E' , then these morphisms are of the form

$$\begin{aligned} \phi(t \otimes v_1) &= av_1 + bv_2 \\ \phi(t \otimes v_2) &= cv_1 - av_2 \\ \psi(v_1^2) &= dv_1 + ev_2 \\ \psi(v_1 v_2) &= -gv_1 - dv_2 \\ \psi(v_2^2) &= fv_1 + gv_2 \\ \xi(v_1 \otimes \tau v_1) &= 0 \\ \xi(v_1 \otimes \tau v_2) &= ht \\ \xi(v_2 \otimes \tau v_1) &= -ht \\ \xi(v_2 \otimes \tau v_2) &= 0. \end{aligned}$$

for $a, b, c, d, e, f, g, h \in B$ satisfying the relations

- (i) $-bg + 2ad + ce = 0$;
- (ii) $-bf + 2ag + cd = 0$;
- (iii) $(a^2 + bc)h = \frac{3}{2}(a(ef - dg) + b(g^2 - df) + c(eg - d^2))$.

3.6. Remark. The morphism ψ is what Miranda calls a *triple cover homomorphism* in [7]. Such a homomorphism induces a triple cover $\text{Spec}_X(\mathcal{O}_X \oplus \mathcal{E}') \rightarrow X$.

3.7. Remark. There are certainly many solutions to the above system of constraints. For instance, $a = b = c = d = g = 1, e = -1, f = 3, h = -6$ and $a = b = c = d = f = 1, e = -2, g = 0, h = -3$ are both possible solutions.

4. GLOBAL ANALYSIS

The previous section obtained a local description of flat S_3 -covers. We now use this local description to obtain a global description. In light of Corollary 3.5, we now expect to characterize such covers by a submodule of

$$\mathrm{Hom}(\mathcal{L} \otimes \mathcal{E}', \mathcal{E}') \oplus \mathrm{Hom}(S^2 \mathcal{E}', \mathcal{E}') \oplus \mathrm{Hom}(\mathcal{E}' \otimes \tau \mathcal{E}', \mathcal{L}).$$

We need a basis-free restatement of Corollary 3.5.

4.1. Definition. — Let $M_1 \leq \mathrm{Hom}(L \otimes E', E')$ denote the submodule consisting of elements ϕ of the form

$$\begin{aligned}\phi(t \otimes v_1) &= av_1 + bv_2 \\ \phi(t \otimes v_2) &= cv_1 - av_2.\end{aligned}$$

Let $M_2 \leq \mathrm{Hom}(S^2 E', E')$ denote the submodule consisting of elements ψ of the form

$$\begin{aligned}\psi(v_1^2) &= dv_1 + ev_2 \\ \psi(v_1 v_2) &= -gv_1 - dv_2 \\ \psi(v_2^2) &= fv_1 + gv_2.\end{aligned}$$

Let $M_3 \leq \mathrm{Hom}(E' \otimes \tau E', L)$ denote the submodule consisting of elements ξ of the form

$$\begin{aligned}\xi(v_1 \otimes \tau v_1) &= 0 \\ \xi(v_1 \otimes \tau v_2) &= ht \\ \xi(v_2 \otimes \tau v_1) &= -ht \\ \xi(v_2 \otimes \tau v_2) &= 0.\end{aligned}$$

4.2. Lemma. — M_1, M_2 , and M_3 are well-defined.

Proof. Let $s = at$ be another generator for L , and $\{w_1, w_2\}$ be another basis for E' , with change of basis matrix

$$C = \begin{bmatrix} \lambda_1 & \mu_1 \\ \lambda_2 & \mu_2 \end{bmatrix}.$$

A straightforward calculation then gives

$$\begin{aligned}
\phi(s \otimes w_1) &= a'w_1 + b'w_2 \\
\phi(s \otimes w_2) &= c'w_1 - a'w_2 \\
\psi(w_1^2) &= d'w_1 + e'w_2 \\
\psi(w_1w_2) &= -g'w_1 - d'w_2 \\
\psi(w_2^2) &= f'w_1 + g'w_2 \\
\xi(w_1 \otimes \tau w_1) &= 0 \\
\xi(w_1 \otimes \tau w_2) &= h's \\
\xi(w_2 \otimes \tau w_1) &= -h's \\
\xi(w_2 \otimes \tau w_2) &= 0,
\end{aligned}$$

where

$$\begin{aligned}
a' &= \frac{a}{\det(C)}(-\lambda_1\lambda_2b + \lambda_1\mu_2a + \lambda_2\mu_1a + \mu_1\mu_2c) \\
b' &= \frac{a}{\det(C)}(\lambda_1^2b - 2\lambda_1\mu_1a - \mu_1^2c) \\
c' &= \frac{a}{\det(C)}(-\lambda_2^2b + 2\lambda_2\mu_2a + \mu_2^2c) \\
d' &= \frac{1}{\det(C)}(-\lambda_1^2\lambda_2e + \lambda_1^2\mu_2d + 2\lambda_1\lambda_2\mu_1d - 2\lambda_1\mu_1\mu_2g - \lambda_2\mu_1^2g + \mu_1^2\mu_2f) \\
e' &= \frac{1}{\det(C)}(\lambda_1^3e - 3\lambda_1^2\mu_1d + 3\lambda_1\mu_1^2g - \mu_1^3f) \\
f' &= \frac{1}{\det(C)}(-\lambda_2^3e + 3\lambda_2^2\mu_2d - 3\lambda_2\mu_2^2g + \mu_2^3f) \\
g' &= \frac{1}{\det(C)}(\lambda_1\lambda_2^2e - 2\lambda_1\lambda_2\mu_2d + \lambda_1\mu_2^2g - \lambda_2^2\mu_1d + 2\lambda_2\mu_1\mu_2g - \mu_1\mu_2^2f) \\
h' &= \frac{\det(C)}{a}h.
\end{aligned}$$

□

4.3. Lemma. — *There exist natural isomorphisms*

- (i) $\mathbf{F}_1 : M_1 \xrightarrow{\sim} \text{Hom}(L \otimes S^2E', \bigwedge^2 E')$;
- (ii) $\mathbf{F}_2 : M_2 \xrightarrow{\sim} \text{Hom}(S^3E', \bigwedge^2 E')$;
- (iii) $\mathbf{F}_3 : M_3 \xrightarrow{\sim} \text{Hom}(\bigwedge^2 E', L)$.

Proof. The proof of (ii) is detailed in [7, Prop. 3.3]. The proofs of (i) and (iii) are similar to that of (ii), and are given here. The method of proof will be used repeatedly.

We begin by proving (iii). Assume $\xi \in M_3$ is of the form of Definition 5.2. The induced morphism

$$E' \otimes E' \xrightarrow{1 \otimes \tau} E' \otimes \tau E' \xrightarrow{\xi} L$$

maps

$$\begin{aligned} v_1 \otimes v_1 &\longmapsto v_1 \otimes \tau v_1 \longmapsto 0 \\ v_1 \otimes v_2 &\longmapsto v_1 \otimes \tau v_2 \longmapsto ht \\ v_2 \otimes v_1 &\longmapsto v_2 \otimes \tau v_1 \longmapsto -ht \\ v_2 \otimes v_2 &\longmapsto v_2 \otimes \tau v_2 \longmapsto 0, \end{aligned}$$

and hence factors through the canonical morphism from $E' \otimes E'$ to $\bigwedge^2 E'$. Denote this induced morphism $\Xi : \bigwedge^2 E' \rightarrow L$, and define $\mathbf{F}_3(\xi) = \Xi$. The inverse morphism, \mathbf{G}_3 , is given by pre-composing an element $\Xi \in \text{Hom}(\bigwedge^2 E', L)$ with the morphism

$$E' \otimes \tau E' \xrightarrow{1 \otimes \tau} E' \otimes E' \xrightarrow{\text{can}} \bigwedge^2 E'.$$

This morphism is defined without reference to a basis, and so is clearly natural. To check it is an inverse of \mathbf{F}_3 , suppose $\Xi \in \text{Hom}(\bigwedge^2 E', L)$ is of the form

$$v_1 \wedge v_2 \longmapsto ht.$$

Then $\mathbf{G}_3(\Xi) \in \text{Hom}(E' \otimes \tau E', L)$ is given by

$$\begin{aligned} v_1 \otimes \tau v_1 &\longmapsto v_1 \otimes v_1 \longmapsto v_1 \wedge v_1 = 0 \longmapsto 0 \\ v_1 \otimes \tau v_2 &\longmapsto v_1 \otimes v_2 \longmapsto v_1 \wedge v_2 \longmapsto ht \\ v_2 \otimes \tau v_1 &\longmapsto v_2 \otimes v_1 \longmapsto v_2 \wedge v_1 \longmapsto -ht \\ v_2 \otimes \tau v_2 &\longmapsto v_2 \otimes v_2 \longmapsto v_2 \wedge v_2 = 0 \longmapsto 0, \end{aligned}$$

and hence equals ξ . This proves (iii).

We next prove (i). Assume $\phi \in M_1$ is in the form of Definition 5.2. The induced morphism

$$L \otimes E' \otimes E' \xrightarrow{\phi \otimes 1} E' \otimes E' \xrightarrow{\text{can}} \bigwedge^2 E'$$

maps

$$\begin{aligned} t \otimes v_1 \otimes v_1 &\longmapsto (av_1 + bv_2) \otimes v_1 \longmapsto -b v_1 \wedge v_2 \\ t \otimes v_1 \otimes v_2 &\longmapsto (av_1 + bv_2) \otimes v_2 \longmapsto a v_1 \wedge v_2 \\ t \otimes v_2 \otimes v_1 &\longmapsto (cv_1 - av_2) \otimes v_1 \longmapsto a v_1 \wedge v_2 \\ t \otimes v_2 \otimes v_2 &\longmapsto (cv_1 - av_2) \otimes v_2 \longmapsto c v_1 \wedge v_2, \end{aligned}$$

and hence factors through the canonical morphism from $L \otimes E' \otimes E'$ to $L \otimes S^2 E'$. Denote this induced morphism $\Phi : L \otimes S^2 E' \rightarrow \bigwedge^2 E'$, and define $\mathbf{F}_1(\phi) = \Phi$.

The inverse morphism is constructed as follows. Observe that we have isomorphisms

$$\begin{aligned}
& \text{Hom}(\text{Hom}(L \otimes S^2 E', \bigwedge^2 E'), \text{Hom}(L \otimes E', E')) \\
& \cong \text{Hom}(L^* \otimes (S^2 E')^* \otimes \bigwedge^2 E', L^* \otimes E'^* \otimes E') \\
& \cong L \otimes S^2 E' \otimes (\bigwedge^2 E')^* \otimes L^* \otimes E'^* \otimes E' \\
& \cong \text{Hom}(L \otimes \bigwedge^2 E' \otimes E', L \otimes E' \otimes S^2 E').
\end{aligned}$$

An element of this final group is the morphism \mathbf{G}_1 defined by

$$l \otimes (e_1 \wedge e_2) \otimes e_3 \longmapsto l \otimes e_1 \otimes e_2 e_3 - l \otimes e_2 \otimes e_1 e_3,$$

for $l \in L, e_i \in E'$. In this form, it is clear that \mathbf{G}_1 does not depend on a choice of basis, and is therefore natural. It remains to check, however, that \mathbf{G}_1 maps $\text{Hom}(L \otimes S^2 E', \bigwedge^2 E')$ isomorphically onto M_1 , and that it is the inverse of the map \mathbf{F}_1 .

We first trace \mathbf{G}_1 backwards through the chain of isomorphisms. With respect to the basis $\{t, v_1, v_2\}$, the morphism \mathbf{G}_1 is given by

$$\begin{aligned}
t \otimes (v_1 \wedge v_2) \otimes v_1 & \longmapsto t \otimes v_1 \otimes v_1 v_2 - t \otimes v_2 \otimes v_1^2 \\
t \otimes (v_1 \wedge v_2) \otimes v_2 & \longmapsto t \otimes v_1 \otimes v_2^2 - t \otimes v_2 \otimes v_1 v_2.
\end{aligned}$$

As an element of $L^* \otimes (\bigwedge^2 E')^* \otimes E'^* \otimes L \otimes E' \otimes S^2 E'$, this morphism corresponds to $t^* \otimes (v_1 \wedge v_2)^* \otimes v_1^* \otimes (t \otimes v_1 \otimes v_1 v_2 - t \otimes v_2 \otimes v_1^2) + t^* \otimes (v_1 \wedge v_2)^* \otimes v_2^* \otimes (t \otimes v_1 \otimes v_2^2 - t \otimes v_2 \otimes v_1 v_2)$.

As an element of $\text{Hom}(L^* \otimes (S^2 E')^* \otimes \bigwedge^2 E', L^* \otimes E'^* \otimes E')$, this element maps

$$\begin{aligned}
t^* \otimes (v_1^2)^* \otimes (v_1 \wedge v_2) & \longmapsto -t^* \otimes v_1^* \otimes v_2 \\
t^* \otimes (v_1 v_2)^* \otimes (v_1 \wedge v_2) & \longmapsto t^* \otimes v_1^* \otimes v_1 - t^* \otimes v_2^* \otimes v_2 \\
t^* \otimes (v_2^2)^* \otimes (v_1 \wedge v_2) & \longmapsto t^* \otimes v_2^* \otimes v_1.
\end{aligned}$$

Now, as an element of $L^* \otimes (S^2 E')^* \otimes \bigwedge^2 E'$, the map Φ corresponds to

$$(-b(t)^* \otimes (v_1^2)^* + a(t)^* \otimes (v_1 v_2)^* + c(t)^* \otimes (v_2^2)^*) \otimes (v_1 \wedge v_2),$$

and hence the image of Φ under \mathbf{G}_1 is

$$bt^* \otimes v_1^* \otimes v_2 + a(t^* \otimes v_1^* \otimes v_1 - t^* \otimes v_2^* \otimes v_2) + ct^* \otimes v_2^* \otimes v_1$$

in $L^* \otimes E'^* \otimes E'$. As an element of $\text{Hom}(L \otimes E', E')$, this element maps

$$\begin{aligned}
t \otimes v_1 & \longmapsto av_1 + bv_2 \\
t \otimes v_2 & \longmapsto cv_1 - av_2,
\end{aligned}$$

and hence equals ϕ . □

4.4. Definition. — Let $S_3 \text{Cov}_B(A) \subset \text{Hom}_B(S^2 A, A)$ denote the submodule of morphisms defining commutative, associative B -algebra structures on A compatible with the given S_3 -action (and hence inducing S_3 -covers $\pi : \text{Spec } A \rightarrow \text{Spec } B$). Let $S_3 \text{Cov}_B(A)^0$ denote those which define integral such algebras.

Also define

$$\text{Build}_B(A) = \text{Hom}(L \otimes S^2 E', \bigwedge^2 E') \oplus \text{Hom}(S^3 E', \bigwedge^2 E') \oplus \text{Hom}(\bigwedge^2 E', L).$$

By the previous lemma and Corollary 3.5, when then have the following:

4.5. Corollary. — *There exists a natural morphism $\mathbf{F} : S_3 \text{Cov}_B(A)^0 \rightarrow \text{Build}_B(A)$.*

As we'll soon see, the morphism \mathbf{F} extracts from an S_3 -cover the “building data” necessary to reconstruct the cover.

Note that any triple $(\Phi, \Psi, \Xi) \in \text{Build}_B(A)$ is of the form

$$\begin{aligned} \Phi(t \otimes v_1^2) &= A(v_1 \wedge v_2) \\ \Phi(t \otimes v_1 v_2) &= B(v_1 \wedge v_2) \\ \Phi(t \otimes v_2^2) &= C(v_1 \wedge v_2) \\ \Psi(v_1^3) &= D(v_1 \wedge v_2) \\ \Psi(v_1^2 v_2) &= E(v_1 \wedge v_2) \\ \Psi(v_1 v_2^2) &= F(v_1 \wedge v_2) \\ \Psi(v_2^3) &= G(v_1 \wedge v_2) \\ \Xi(v_1 \wedge v_2) &= ht \end{aligned}$$

with respect to the basis $\{t, v_1, v_2\}$ for $L \oplus E'$. We use capital letters here to avoid confusion with our notation for a triple $(\phi, \psi, \xi) \in \text{Hom}(L \otimes E', E') \oplus \text{Hom}(S^2 E', E') \oplus \text{Hom}(E' \otimes \tau E', L)$. (From the context, it should be clear when B refers to a constant and when it refers to the ring $\mathcal{O}_X(U)$.) Under the natural isomorphisms of Lemma 4.3, if we let $(\phi, \psi, \xi) = \mathbf{F}^{-1}(\Phi, \Psi, \Xi)$, we have

$$\begin{aligned} \phi(t \otimes v_1) &= Bv_1 - Av_2 \\ \phi(t \otimes v_2) &= Cv_1 - Bv_2 \\ \psi(v_1^2) &= Ev_1 - Dv_2 \\ \psi(v_1 v_2) &= Fv_1 - Ev_2 \\ \psi(v_2^2) &= Gv_1 - Fv_2 \\ \xi(v_1 \otimes \tau v_2) &= ht. \end{aligned}$$

The correspondence with our earlier notation is thus $A = -b, B = a, C = c, D = -e, E = d, F = -g, G = f$. Using this dictionary, the three conditions of Corollary 3.5 become the following:

- (i) $AF - 2BE + CD = 0$;
- (ii) $AG - 2BF + CE = 0$;
- (iii) $h(B^2 - AC) = \frac{3}{2}(B(EF - DG) - A(F^2 - EG) + C(DF - E^2))$.

4.6. Lemma. — *There exists a natural morphism, \mathbf{A}_1 , from $\text{Hom}(L \otimes S^2 E', \bigwedge^2 E') \oplus \text{Hom}(S^3 E', \bigwedge^2 E')$ to $\text{Hom}(L \otimes (\bigwedge^2 E')^{\otimes 2} \otimes E', (\bigwedge^2 E')^{\otimes 2})$ whose kernel consists of precisely those pairs (Φ, Ψ) satisfying conditions (i) and (ii).*

Proof. First, consider the morphism $f_1 : (\bigwedge^2 E')^{\otimes 2} \rightarrow (S^2 E')^{\otimes 2}$ defined by

$$(e_1 \wedge e_2) \otimes (e_3 \wedge e_4) \mapsto e_1 e_3 \otimes e_2 e_4 - e_1 e_4 \otimes e_2 e_3 - e_2 e_3 \otimes e_1 e_4 + e_2 e_4 \otimes e_1 e_3,$$

and the canonical morphism $f_2 : S^2 E' \otimes E' \rightarrow S^3 E'$. Given any pair $(\Phi, \Psi) \in \text{Hom}(L \otimes S^2 E', \bigwedge^2 E') \oplus \text{Hom}(S^3 E', \bigwedge^2 E')$, we then have an induced morphism

$$L \otimes (\bigwedge^2 E')^{\otimes 2} \otimes E' \xrightarrow{1 \otimes f_1 \otimes 1} L \otimes (S^2 E')^{\otimes 2} \otimes E' \xrightarrow{1 \otimes 1 \otimes f_2} L \otimes S^2 E' \otimes S^3 E' \xrightarrow{\Phi \otimes \Psi} (\bigwedge^2 E')^{\otimes 2}$$

Let $\mathbf{A}_1(\Phi, \Psi)$ denote this morphism. With respect to the basis, this morphism is defined by

$$\begin{aligned} t \otimes (v_1 \wedge v_2) \otimes (v_1 \wedge v_2) \otimes v_1 &\mapsto t \otimes (v_1^2 \otimes v_2^2 - 2(v_1 v_2 \otimes v_1 v_2) + v_2^2 \otimes v_1^2) \otimes v_1 \\ &\mapsto t \otimes (v_1^2 \otimes v_1 v_2^2 - 2(v_1 v_2 \otimes v_1^2 v_2) + v_2^2 \otimes v_1^3) \\ &\mapsto (AF - 2BE + CD)(v_1 \wedge v_2)^{\otimes 2} \\ t \otimes (v_1 \wedge v_2) \otimes (v_1 \wedge v_2) \otimes v_2 &\mapsto t \otimes (v_1^2 \otimes v_2^2 - 2(v_1 v_2 \otimes v_1 v_2) + v_2^2 \otimes v_1^2) \otimes v_2 \\ &\mapsto t \otimes (v_1^2 \otimes v_2^3 - 2(v_1 v_2 \otimes v_1 v_2^2) + v_2^2 \otimes v_1^2 v_2) \\ &\mapsto (AG - 2BF + CE)(v_1 \wedge v_2)^{\otimes 2} \end{aligned}$$

Thus, conditions (i) and (ii) together are equivalent to the morphism $\mathbf{A}_1(\Phi, \Psi)$ being the zero map. \square

For the proof of the following lemma, note that given any pair $(\Phi, \Psi) \in \text{Hom}(L \otimes S^2 E', \bigwedge^2 E') \oplus \text{Hom}(S^3 E', \bigwedge^2 E')$, we naturally have induced maps $\bigwedge^2(\phi), \bigwedge^2(\psi)$. In terms of the basis, these are given by

$$(t \otimes v_1) \wedge (t \otimes v_2) \mapsto -(B^2 - AC)(v_1 \wedge v_2)$$

and

$$\begin{aligned} v_1^2 \wedge v_1 v_2 &\mapsto (DF - E^2)(v_1 \wedge v_2) \\ v_1^2 \wedge v_2^2 &\mapsto (DG - EF)(v_1 \wedge v_2) \\ v_1 v_2 \wedge v_2^2 &\mapsto (EG - F^2)(v_1 \wedge v_2), \end{aligned}$$

respectively.

4.7. Lemma. — *There exists a natural morphism, \mathbf{A}_2 , from $\text{Build}_B(A)$ to $\text{Hom}(L \otimes (\bigwedge^2 E')^{\otimes 3}, (\bigwedge^2 E')^{\otimes 2})$ whose kernel consists of precisely those triples (Φ, Ψ, Ξ) satisfying condition (iii).*

Proof. Consider the two natural morphisms, $f_1 : (\bigwedge^2 E')^{\otimes 3} \rightarrow \bigwedge^3(S^2 E')$ and $f_2 : \bigwedge^3(S^2 E') \rightarrow S^2 E' \otimes \bigwedge^2(S^2 E')$, defined by

$$(e_1 \wedge e_2) \otimes (e_3 \wedge e_4) \otimes (e_5 \wedge e_6) \mapsto e_1^2 \wedge e_2^2 \wedge (e_3 e_5 - e_3 e_6 - e_4 e_5 + e_4 e_6),$$

and

$$e_1 e_2 \wedge e_3 e_4 \wedge e_5 e_6 \longmapsto \frac{3}{4} (e_1 e_2 \otimes (e_3 e_4 \wedge e_5 e_6) - e_3 e_4 \otimes (e_1 e_2 \wedge e_5 e_6) + e_5 e_6 \otimes (e_1 e_2 \wedge e_3 e_4)),$$

respectively. Given any pair $(\Phi, \Psi) \in \text{Hom}(L \otimes S^2 E', \bigwedge^2 E') \oplus \text{Hom}(S^3 E', \bigwedge^2 E')$ we then have an induced morphism

$$L \otimes (\bigwedge^2 E')^{\otimes 3} \xrightarrow{1 \otimes f_1} L \otimes \bigwedge^3(S^2 E') \xrightarrow{1 \otimes f_2} L \otimes S^2 E' \otimes \bigwedge^2(S^2 E') \xrightarrow{\Phi \otimes \bigwedge^2(\psi)} (\bigwedge^2 E')^{\otimes 2}.$$

Denote this morphism by $\mathbf{A}_{2,1}(\Phi, \Psi)$. In terms of the basis, this morphism is given by

$$\begin{aligned} t \otimes (v_1 \wedge v_2)^{\otimes 3} &\longmapsto t \otimes (v_1^2 \wedge v_2^2 \wedge (v_1^2 - v_1 v_2 - v_2 v_1 + v_2^2)) = 2t \otimes (v_1^2 \wedge v_1 v_2 \wedge v_2^2) \\ &\longmapsto \frac{3}{2} t \otimes (v_1^2 \otimes (v_1 v_2 \wedge v_2^2) - v_1 v_2 \otimes (v_1^2 \wedge v_2^2) + v_2^2 \otimes (v_1^2 \wedge v_1 v_2)) \\ &\longmapsto \frac{3}{2} (A(EG - F^2) - B(DG - EF) + C(DF - E^2)) (v_1 \wedge v_2)^{\otimes 2} \\ &= \frac{3}{2} (B(EF - DG) - A(F^2 - EG) + C(DF - E^2)) (v_1 \wedge v_2)^{\otimes 2}. \end{aligned}$$

Given any pair $(\Phi, \Xi) \in \text{Hom}(L \otimes S^2 E', \bigwedge^2 E') \oplus \text{Hom}(\bigwedge^2 E', L)$, we have an induced morphism

$$\bigwedge^2 E' \otimes E' \otimes L \otimes E' \xrightarrow{\Xi \otimes 1 \otimes 3} L \otimes E' \otimes L \otimes E' \xrightarrow{\text{can}} \bigwedge^2(L \otimes E') \xrightarrow{\bigwedge^2(\phi)} \bigwedge^2 E'.$$

In terms of the basis, this morphism maps

$$\begin{aligned} t \otimes (v_1 \wedge v_2) \otimes v_1 \otimes v_1 &\longmapsto ht \otimes v_1 \otimes t \otimes v_1 \longmapsto h(t \otimes v_1) \wedge (t \otimes v_1) = 0 \longmapsto 0 \\ t \otimes (v_1 \wedge v_2) \otimes v_1 \otimes v_2 &\longmapsto ht \otimes v_1 \otimes t \otimes v_2 \longmapsto h(t \otimes v_1) \wedge (t \otimes v_2) \longmapsto -h(B^2 - AC)(v_1 \wedge v_2) \\ t \otimes (v_1 \wedge v_2) \otimes v_2 \otimes v_1 &\longmapsto ht \otimes v_2 \otimes t \otimes v_1 \longmapsto h(t \otimes v_2) \wedge (t \otimes v_1) \longmapsto h(B^2 - AC)(v_1 \wedge v_2) \\ t \otimes (v_1 \wedge v_2) \otimes v_2 \otimes v_2 &\longmapsto ht \otimes v_2 \otimes t \otimes v_2 \longmapsto h(t \otimes v_2) \wedge (t \otimes v_2) = 0 \longmapsto 0, \end{aligned}$$

and hence factors through the canonical morphism (induced from) $E' \otimes E' \rightarrow \bigwedge^2 E'$. This gives a morphism

$$\mathbf{A}'_{2,2}(\Phi, \Xi) : L \otimes (\bigwedge^2 E')^{\otimes 2} \rightarrow \bigwedge^2 E'.$$

Let $\mathbf{A}_{2,2}(\Phi, \Xi)$ then denote the induced morphism

$$L \otimes (\bigwedge^2 E')^{\otimes 3} \xrightarrow{\mathbf{A}'_{2,2} \otimes 1} (\bigwedge^2 E')^{\otimes 2}.$$

In terms of the basis, this morphism is given by

$$t \otimes (v_1 \wedge v_2)^{\otimes 3} \longmapsto -h(B^2 - AC)(v_1 \wedge v_2)^{\otimes 2}.$$

Lastly, define $\mathbf{A}_2(\Phi, \Psi, \Xi) \in \text{Hom}(L \otimes (\bigwedge^2 E')^{\otimes 3}, (\bigwedge^2 E')^{\otimes 2})$ as the sum $\mathbf{A}_{2,1}(\Phi, \Psi) + \mathbf{A}_{2,2}(\Phi, \Xi)$. In terms of the basis, this morphism is given by

$$t \otimes (v_1 \wedge v_2)^{\otimes 3} \longmapsto \left(\frac{3}{2} (B(EF - DG) - A(F^2 - EG) + C(DF - E^2)) - h(B^2 - AC) \right) (v_1 \wedge v_2)^{\otimes 2},$$

and hence condition (iii) is equivalent to the vanishing of $\mathbf{A}_2(\Phi, \Psi, \Xi)$. \square

4.8. Definition. — For notational simplicity, let us define

$$\text{Compat}_B(A) = \text{Hom}(L \otimes (\bigwedge^2 E')^{\otimes 2} \otimes E', (\bigwedge^2 E')^{\otimes 2}) \oplus \text{Hom}(L \otimes (\bigwedge^2 E')^{\otimes 3}, (\bigwedge^2 E')^{\otimes 2}).$$

The previous two lemmas then give the following:

4.9. Corollary. — *There is a natural morphism $\mathbf{A} : \text{Build}_B(A) \rightarrow \text{Compat}_B(A)$ whose kernel consists of precisely the morphisms satisfying conditions (i)-(iii).*

In other words, the morphism \mathbf{A} tests the building data for the compatibility conditions (arising from associativity constraints) necessary for the data to induce an S_3 -cover.

We next recover from any building data the three morphisms defining an algebra structure:

$$\alpha : S^2 L \rightarrow B, \quad \beta : L \otimes E \rightarrow E, \quad \gamma : S^2 E \rightarrow A.$$

4.10. Lemma. — *There exists a natural morphism, \mathbf{B}_1 , from $\text{Hom}(L \otimes S^2 E', \bigwedge^2 E')$ to $\text{Hom}(S^2 L, B)$, taking an element Φ of the form*

$$\begin{aligned} \Phi(t \otimes v_1^2) &= A(v_1 \wedge v_2) \\ \Phi(t \otimes v_1 v_2) &= B(v_1 \wedge v_2) \\ \Phi(t \otimes v_2^2) &= C(v_1 \wedge v_2) \end{aligned}$$

to an element α of the form

$$\alpha(t^2) = -3(B^2 - AC).$$

Proof. Observe that we have isomorphisms

$$\begin{aligned} \text{Hom} \left(\text{Hom} \left(\bigwedge^2 (L \otimes E'), \bigwedge^2 E' \right), \text{Hom}(S^2 L, B) \right) &\cong \text{Hom} \left(\left(\bigwedge^2 (L \otimes E') \right)^* \otimes \bigwedge^2 E', (S^2 L)^* \right) \\ &\cong \text{Hom} \left(\bigwedge^2 E' \otimes S^2 L, \bigwedge^2 (L \otimes E') \right). \end{aligned}$$

An element of this last group is the morphism $\tilde{\mathbf{B}}_1$ defined by

$$(e_1 \wedge e_2) \otimes l_1 l_2 \mapsto \frac{3}{2} ((l_1 \otimes e_1) \wedge (l_2 \otimes e_2) - (l_1 \otimes e_2) \wedge (l_2 \otimes e_1)).$$

In terms of the basis, this morphism is defined by

$$(v_1 \wedge v_2) \otimes t^2 \mapsto \frac{3}{2} ((t \otimes v_1) \wedge (t \otimes v_2) - (t \otimes v_2) \wedge (t \otimes v_1)) = 3(t \otimes v_1) \wedge (t \otimes v_2).$$

This morphism, considered as an element of $\text{Hom}((\bigwedge^2 (L \otimes E'))^* \otimes \bigwedge^2 E', (S^2 L)^*)$, is given by

$$((t \otimes v_1) \wedge (t \otimes v_2))^* \otimes (v_1 \wedge v_2) \mapsto 3(t^2)^*.$$

The morphism $\bigwedge^2(\phi)$, when considered as an element of $(\bigwedge^2(L \otimes E'))^* \otimes \bigwedge^2 E'$, is

$$-(B^2 - AC)((t \otimes v_1) \wedge (t \otimes v_2))^* \otimes (v_1 \wedge v_2).$$

Thus, $\tilde{\mathbf{B}}_1$ maps $\bigwedge^2(\phi)$ to the element

$$-3(B^2 - AC)(t^2)^*,$$

which corresponds to the map

$$t^2 \longmapsto -3(B^2 - AC).$$

Thus, the composition $\mathbf{B}_1 = \tilde{\mathbf{B}}_1 \circ \bigwedge^2 \circ \mathbf{F}_1^{-1}$ is the desired natural transformation. \square

4.11. Lemma. — *There exists a natural morphism, \mathbf{B}_2 , from $\text{Hom}(L \otimes S^2 E', \bigwedge^2 E')$ to $\text{Hom}(L \otimes E, E)$, taking an element Φ of the form*

$$\begin{aligned}\Phi(t \otimes v_1^2) &= A(v_1 \wedge v_2) \\ \Phi(t \otimes v_1 v_2) &= B(v_1 \wedge v_2) \\ \Phi(t \otimes v_2^2) &= C(v_1 \wedge v_2)\end{aligned}$$

to an element β of the form

$$\begin{aligned}\beta(t \otimes v_1) &= Bv_1 - Av_2 - 2B\tau v_1 + 2A\tau v_2 \\ \beta(t \otimes v_2) &= Cv_1 - Bv_2 - 2C\tau v_1 + 2B\tau v_2 \\ \beta(t \otimes \tau v_1) &= 2Bv_1 - 2Av_2 - B\tau v_1 + A\tau v_2 \\ \beta(t \otimes \tau v_2) &= 2Cv_1 - 2Bv_2 - C\tau v_1 + B\tau v_2.\end{aligned}$$

Proof. Observe that $\phi = \mathbf{F}_1^{-1}(\Phi)$ induces a morphism

$$L \otimes E' \xrightarrow{\phi} E' \xrightarrow{\tau} \tau E' \subset E,$$

and hence two morphisms:

$$\beta_1 : L \otimes E' \xrightarrow{(\phi, -2(\tau \circ \phi))} E' \oplus \tau E' = E,$$

and

$$\beta_2 : L \otimes \tau E' \xrightarrow{\tau \otimes \tau} L \otimes E' \xrightarrow{(\phi, -2(\tau \circ \phi))} E' \oplus \tau E' = E \xrightarrow{\tau} E.$$

Let $\mathbf{B}_2(\Phi) = \langle \beta_1, \beta_2 \rangle : L \otimes E \cong (L \otimes E') \oplus (L \otimes \tau E') \rightarrow E$. In terms of the basis, this morphism is given by

$$\begin{aligned}t \otimes v_1 &\longmapsto Bv_1 - Av_2 - 2B\tau v_1 + 2A\tau v_2 \\ t \otimes v_2 &\longmapsto Cv_1 - Bv_2 - 2C\tau v_1 + 2B\tau v_2 \\ t \otimes \tau v_1 &\longmapsto 2Bv_1 - 2Av_2 - B\tau v_1 + A\tau v_2 \\ t \otimes \tau v_2 &\longmapsto 2Cv_1 - 2Bv_2 - C\tau v_1 + B\tau v_2.\end{aligned}$$

\square

4.12. Lemma. — *There exists a natural morphism, \mathbf{B}_3 , from $\text{Hom}(S^3 E', \bigwedge^2 E') \oplus \text{Hom}(\bigwedge^2 E', L)$ to $\text{Hom}(S^2 E, A)$, taking a pair (Ψ, Ξ) of the form*

$$\begin{aligned}\Psi(v_1^3) &= D(v_1 \wedge v_2) \\ \Psi(v_1^2 v_2) &= E(v_1 \wedge v_2) \\ \Psi(v_1 v_2^2) &= F(v_1 \wedge v_2) \\ \Psi(v_2^3) &= G(v_1 \wedge v_2) \\ \Xi(v_1 \wedge v_2) &= ht\end{aligned}$$

to an element γ of the form

$$\begin{aligned}\gamma(v_1^2) &= 6(E^2 - DF) + Ev_1 - Dv_2 - 2E\tau v_1 + 2D\tau v_2 \\ \gamma(v_1 v_2) &= 3(EF - DG) + Fv_1 - Ev_2 - 2F\tau v_1 + 2E\tau v_2 \\ \gamma(v_2^2) &= 6(F^2 - EG) + Gv_1 - Fv_2 - 2G\tau v_1 + 2F\tau v_2 \\ \gamma(v_1 \cdot \tau v_1) &= 3(E^2 - DF) - Ev_1 + Dv_2 - E\tau v_1 + D\tau v_2 \\ \gamma(v_1 \cdot \tau v_2) &= \frac{3}{2}(EF - DG) + ht - Fv_1 + Ev_2 - F\tau v_1 + E\tau v_2 \\ \gamma(v_2 \cdot \tau v_1) &= \frac{3}{2}(EF - DG) - ht - Fv_1 + Ev_2 - F\tau v_1 + E\tau v_2 \\ \gamma(v_2 \cdot \tau v_2) &= 3(F^2 - EG) - Gv_1 + Fv_2 - G\tau v_1 + F\tau v_2 \\ \gamma((\tau v_1)^2) &= 6(E^2 - DF) - 2Ev_1 + 2Dv_2 + E\tau v_1 - D\tau v_2 \\ \gamma((\tau v_1)(\tau v_2)) &= 3(EF - DG) - 2Fv_1 + 2Ev_2 + F\tau v_1 - E\tau v_2 \\ \gamma((\tau v_2)^2) &= 6(F^2 - EG) - 2Gv_1 + 2Fv_2 + G\tau v_1 - F\tau v_2.\end{aligned}$$

Proof. Observe that we have isomorphisms

$$\begin{aligned}\text{Hom}\left(\text{Hom}\left(\bigwedge^2(S^2 E'), \bigwedge^2 E'\right), \text{Hom}(S^2 E', B)\right) &\cong \text{Hom}\left(\left(\bigwedge^2(S^2 E')\right)^* \otimes \bigwedge^2 E', (S^2 E')^*\right) \\ &\cong \text{Hom}\left(\left(\bigwedge^2 E'\right) \otimes S^2 E', \bigwedge^2(S^2 E')\right).\end{aligned}$$

A natural element of this last group is the morphism $\tilde{\mathbf{B}}_{3,1}$ defined by

$$(e_1 \wedge e_2) \otimes e_3 e_4 \longmapsto -3(e_1 e_3 \wedge e_2 e_4 + e_1 e_4 \wedge e_2 e_3).$$

In terms of the basis, this morphism is given by

$$\begin{aligned}(v_1 \wedge v_2) \otimes v_1^2 &\longmapsto -3(v_1^2 \wedge v_1 v_2 + v_1^2 \wedge v_1 v_2) = -6(v_1^2 \wedge v_1 v_2) \\ (v_1 \wedge v_2) \otimes v_1 v_2 &\longmapsto -3(v_1^2 \wedge v_2^2 + v_1 v_2 \wedge v_1 v_2) = -3(v_1^2 \wedge v_2^2) \\ (v_1 \wedge v_2) \otimes v_2^2 &\longmapsto -3(v_1 v_2 \wedge v_2^2 + v_1 v_2 \wedge v_2^2) = -6(v_1 v_2 \wedge v_2^2).\end{aligned}$$

As an element of $\text{Hom}((\wedge^2(S^2E'))^* \otimes \wedge^2 E', (S^2E')^*)$, this corresponds to the map

$$\begin{aligned} (v_1^2 \wedge v_1 v_2)^* \otimes (v_1 \wedge v_2) &\longmapsto -6(v_1^2)^* \\ (v_1^2 \wedge v_2^2)^* \otimes (v_1 \wedge v_2) &\longmapsto -3(v_1 v_2)^* \\ (v_1 v_2 \wedge v_2^2)^* \otimes (v_1 \wedge v_2) &\longmapsto -6(v_2^2)^*. \end{aligned}$$

The morphism $\wedge^2(\psi)$, as an element of $(\wedge^2(S^2E'))^* \otimes \wedge^2 E'$, is given by

$$(DF - E^2)(v_1^2 \wedge v_1 v_2)^* \otimes (v_1 \wedge v_2) + (DG - EF)(v_1^2 \wedge v_2^2)^* \otimes (v_1 \wedge v_2) + (EG - F^2)(v_1 v_2 \wedge v_2^2)^* \otimes (v_1 \wedge v_2).$$

Thus, $\tilde{\mathbf{B}}_{3,1}$ maps $\wedge^2(\psi)$ to the element

$$6(E^2 - DF)(v_1^2)^* + 3(EF - DG)(v_1 v_2)^* + 6(F^2 - EG)(v_2^2)^*,$$

which corresponds to the morphism $\tilde{\gamma}_1$ defined by

$$\begin{aligned} v_1^2 &\longmapsto 6(E^2 - DF) \\ v_1 v_2 &\longmapsto 3(EF - DG) \\ v_2^2 &\longmapsto 6(F^2 - EG). \end{aligned}$$

This induces morphisms

$$\begin{aligned} \gamma'_{1,1} : E' \otimes E' &\xrightarrow{\text{can}} S^2 E' \xrightarrow{\tilde{\gamma}_1} B \\ \gamma'_{1,2} : E' \otimes \tau E' &\xrightarrow{1 \otimes \tau} E' \otimes E' \xrightarrow{\text{can}} S^2 E' \xrightarrow{\frac{1}{2}\tilde{\gamma}_1} B \\ \gamma'_{1,3} : \tau E' \otimes E' &\xrightarrow{\tau \otimes 1} E' \otimes E' \xrightarrow{\text{can}} S^2 E' \xrightarrow{\frac{1}{2}\tilde{\gamma}_1} B \\ \gamma'_{1,4} : \tau E' \otimes \tau E' &\xrightarrow{\tau \otimes \tau} E' \otimes E' \xrightarrow{\text{can}} S^2 E' \xrightarrow{\tilde{\gamma}_1} B, \end{aligned}$$

which together define a morphism $\gamma'_1 : E \otimes E \cong (E' \otimes E') \oplus (E' \otimes \tau E') \oplus (\tau E' \otimes E') \oplus (\tau E' \otimes \tau E') \rightarrow B$. By construction, this morphism factors through the canonical morphism from $E \otimes E$ to $S^2 E$, and gives the first-coordinate morphism $\gamma_1 : S^2 E \rightarrow B$. Let $\mathbf{B}_{3,1}(\Psi) = \gamma_1$.

Next, observe that $\xi = \mathbf{F}_3^{-1}(\Xi)$ induces a morphism

$$\xi' : \tau E' \otimes E' \xrightarrow{\text{can}} E' \otimes \tau E' \xrightarrow{\xi} L.$$

Define $\gamma'_2 : E \otimes E \rightarrow L$ by $\gamma'_2 = \langle 0, \xi, \xi', 0 \rangle$. By construction, this morphism factors through the canonical morphism from $E \otimes E$ to $S^2 E$, and gives the second-coordinate morphism $\gamma_2 : S^2 E \rightarrow L$. Let $\mathbf{B}_{3,2}(\Xi) = \gamma_2$.

Lastly, observe that the morphism $\psi = \mathbf{F}_2^{-1}(\Psi)$ induces morphisms

$$\begin{aligned}\gamma'_{3,1} : E' \otimes E' &\xrightarrow{\text{can}} S^2 E' \xrightarrow{\psi-2(\tau \circ \psi)} E' \oplus \tau E' \\ \gamma'_{3,2} : E' \otimes \tau E' &\xrightarrow{1 \otimes \tau} E' \otimes E' \xrightarrow{\text{can}} S^2 E' \xrightarrow{-\psi-(\tau \circ \psi)} E' \oplus \tau E' \\ \gamma'_{3,3} : \tau E' \otimes E' &\xrightarrow{\tau \otimes 1} E' \otimes E' \xrightarrow{\text{can}} S^2 E' \xrightarrow{-\psi-(\tau \circ \psi)} E' \oplus \tau E' \\ \gamma'_{3,4} : \tau E' \otimes \tau E' &\xrightarrow{\tau \otimes \tau} E' \otimes E' \xrightarrow{\text{can}} S^2 E' \xrightarrow{-2\psi+(\tau \circ \psi)} E' \oplus \tau E'.\end{aligned}$$

These together define a morphism $\gamma'_3 : E \otimes E \rightarrow E$ by $\gamma'_3 = \langle \gamma'_{3,1}, \gamma'_{3,2}, \gamma'_{3,3}, \gamma'_{3,4} \rangle$. By construction, this factors through the canonical morphism from $E \otimes E$ to $S^2 E$, and gives the third-coordinate morphism $\gamma_3 : S^2 E \rightarrow L$. Let $\mathbf{B}_{3,3}(\Psi) = \gamma_3$. \square

4.13. Corollary. — *There is a natural morphism $\mathbf{B} : \text{Build}_B(A) \rightarrow \text{Hom}_B(S^2 A, A)$ which maps $\ker \mathbf{A} \subset \text{Build}_B(A)$ into $S_3 \text{Cov}_B(A)$.*

In other words, given compatible building data, \mathbf{B} builds the associated S_3 -cover.

By naturality, the results above sheafify to give the following:

4.14. Main Theorem. — *Let X be an integral, Noetherian R -scheme. Suppose \mathcal{L} is an invertible \mathcal{O}_X -module on which S_3 acts via the sign character, and \mathcal{E} is a locally free \mathcal{O}_X -module of rank 4 together with an S_3 -action such that (under the induced $R[S_3]$ -action) e_i acts as $\delta_{3i} \text{Id}_{\mathcal{E}}$. Let $\mathcal{E} = \mathcal{E}' \oplus \tau \mathcal{E}'$ be the \mathcal{O}_X -module decomposition of \mathcal{E} induced by $e_3 = e_{31} + e_{32}$, and let $\mathcal{A} = \mathcal{O}_X \oplus \mathcal{L} \oplus \mathcal{E}$.*

There exist morphisms $\mathbf{F} : S_3 \text{Cov}_X(\mathcal{A})^0 \rightarrow \text{Build}_X(\mathcal{A})$, $\mathbf{B} : \text{Build}_X(\mathcal{A}) \rightarrow \text{Hom}_X(S^2 \mathcal{A}, \mathcal{A})$, and $\mathbf{A} : \text{Build}_X(\mathcal{A}) \rightarrow \text{Compat}_X(\mathcal{A})$ with the following properties:

- (i) *The morphism \mathbf{F} factors through $\ker \mathbf{A}$; i.e., if \mathcal{A} is endowed with an algebra structure (compatible with the given S_3 -action) such that $\pi : \mathbf{Spec}_X \mathcal{A} \rightarrow X$ is a flat S_3 -cover of integral, Noetherian R -schemes, then the algebra structure on \mathcal{A} defines an element $(\alpha, \beta, \gamma) \in S_3 \text{Cov}_X(\mathcal{A})^0$ with $\mathbf{F}(\alpha, \beta, \gamma) \in \ker \mathbf{A}$; and*
- (ii) *The morphism $\mathbf{B}|_{\ker \mathbf{A}}$ factors through $S_3 \text{Cov}_X(\mathcal{A})$; i.e., given any element $(\Phi, \Psi, \Xi) \in \ker \mathbf{A}$, the element $\mathbf{B}(\Phi, \Psi, \Xi)$ defines a commutative, associative \mathcal{O}_X -algebra structure on \mathcal{A} compatible with the given S_3 -action, and hence defines a flat S_3 -cover $\pi : \mathbf{Spec}_X \mathcal{A} \rightarrow X$.*

4.15. Closing remarks. This result represents only the starting point of the study of S_3 -covers. In analogy with the abelian case, an immediate question is under what circumstances the morphisms \mathbf{F} and \mathbf{B} are mutually inverse; that is, when does the building data uniquely determine the cover? Other points of study include (but are certainly not

limited to) the global description of the branch locus, the location and nature of singularities of the covering scheme, and the relationship between the (cohomological) invariants of the base scheme and those of the covering scheme.

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